Modeling of Stadium Waves

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1 Introduction

The wave is a common phenomenon at stadiums during sports events. During Michigan football season, it is hard to attend a game without participating in the wave around the Big House at least once. Inspired by this experience, this project aims to model the behavior of people in the stadium doing the wave. Our model follows the basic concepts of an agent-based, directed Social Network Model where people are modeled as nodes and have certain levels of influence from other nodes. Our motivation is to model human behavior and influence when there is a large number of people in a confined space.The question we aim to answer is: What initial number of people and in what arrangement does it take for the entire stadium to be participating? We define the entire stadium as participating as everyone in a column standing up while the wave is traveling in a complete round back to its initial position. In 2002, a study done at the Eotvos University in Budapest modeled the reaction of a stadium crowd doing the wave [FHV02]. The main ideas from the study we are building on are (1) the wave usually rolls in a clockwise direction (2) the wave travels at about 20 seats per second (3) the activation threshold of a person is based on the weighted concentration of active people within a certain radius around them [FHV02].

2 Methodology

2.1 Assumptions, Variables, and Parameters

In the development of the model, several assumptions were made to simplify the modeling process. These are:

1. The stadium is at full capacity
2. The wave travels in the clockwise direction
3. People are equidistant from each other
4. Everyone is equally likely to participate in the wave

We are assuming a stadium at full capacity in order to simplify the model to avoid any physical gaps in the wave. Assuming the wave travels in the clockwise direction is based on the 2002 study[1] . This assumption allows us to choose the influence factor accordingly, i.e. people to the right will have a higher influence than people to the left. Assuming people are equidistant from each other simplifies the model by allowing us to assign the influence factor based on number of people away instead of distance away. The model is further simplified by the assumption that the stadium has an equal number of people in every row as opposed to a traditional stadium with a decreasing number of people in each row as the stadium tapers down.

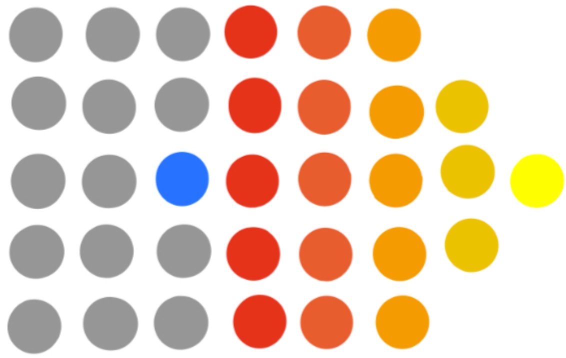


Figure 1: Distribution of weights influencing a spectator (blue)

Due to the lack of research on the subject, many of the parameters were arbitrarily chosen in order to meet the few constraints found in papers like [Gre16], [Gre16]. The stadium is constructed with a set number of rows R and a set number of people per row N. Thus, the entire stadium has a total of NR spectators. These parameters were set to model a scaled-down version of the University of Michigan's stadium "The Big House", that has a capacity of approximately 100,000 people in approximately 100 rows. For ease of computation and visualization, we picked $R = 50$ to be half of the Big House, and $N = 250$ people per row, about 1/4 of the Big House. While these numbers are arbitrary, the model can be scaled to fit the dimensions of any stadium/arena.

A particular spectator will stand up if enough people directly to its right are participating in a wave at any given time. The degree of influence of each of these persons have on the spectator depends on the distance away from the spectator as well as their position relative to it. An spectator sitting in the row r is influenced by the next $p = 5$ persons sitting directly to their right on their row. Additionally, the $l = 2$ rows above and below the spectator will also influence whether they participate in the wave or not. The number of people with non-zero influence in these other rows decreases by one for each row of difference. That is, the rows $r - 1$ and $r + 1$ will have $p - 1$ persons with non-zero influence, the rows $r - 2$ and $r + 2$ next $p - 2$, and so on. For people sitting at the first and last rows, only people sitting on the rows directly above (for the first row) and below (for the last row) are considered. Additionally, the degree of influence a person has on a given spectator decreases linearly from a value of $w_{max} = 1$ to $w_{min} = 0.5$ over a given row. These values are the same for all rows that influence a particular spectator. The arrangement of influences can be seen in Figure 1. The constant W represents the number of people that influence a given spectator. This number is computed by

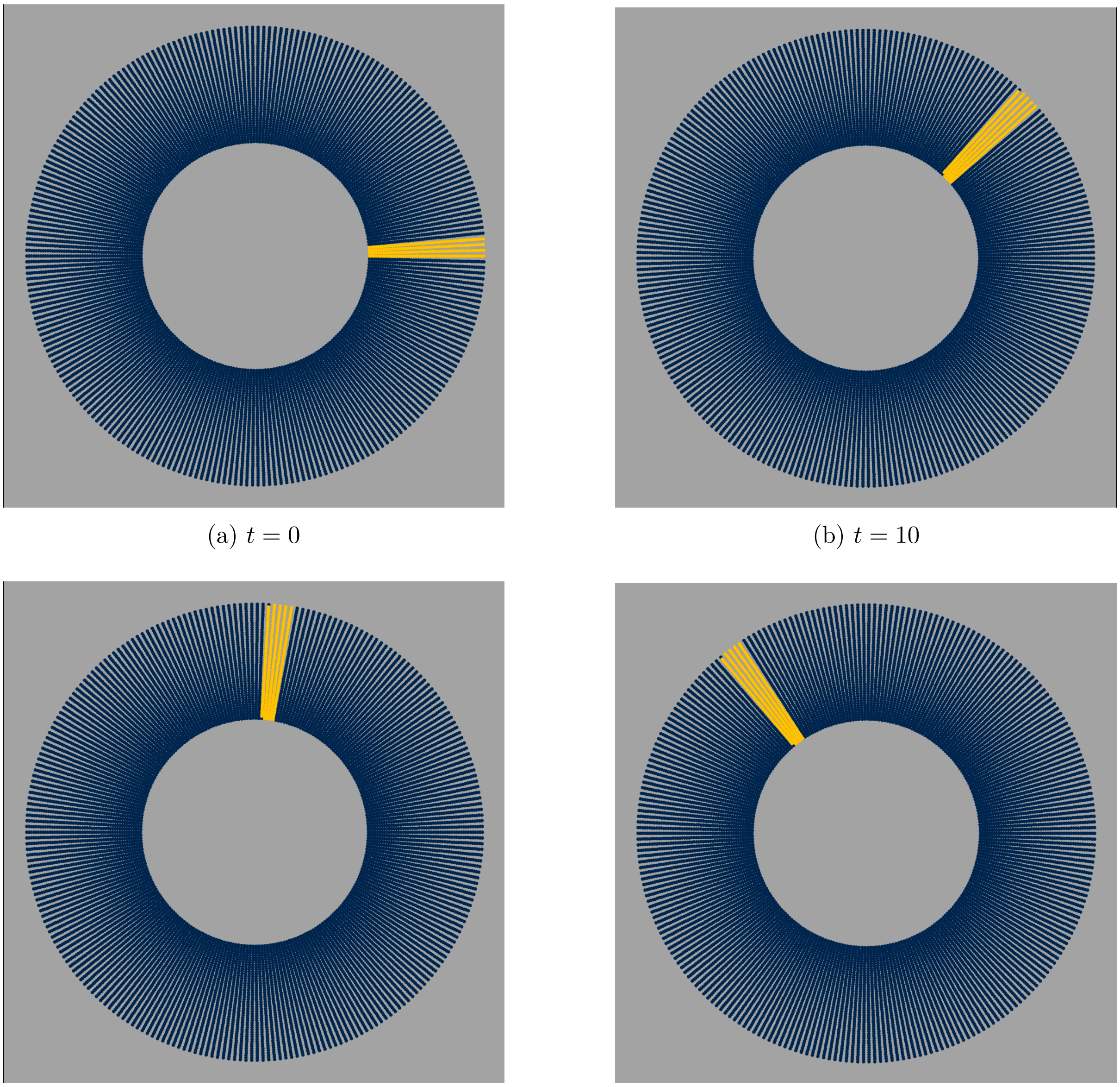
$$W = p + 2(p - 1) + 2(p - 2) + \cdots 2(p - l). \tag{1}$$

Finally, the parameter h indicates how likely a given spectator is to stand up and participate in the wave. This parameter was set to $h = 0.2$. Further discussion about the choice of parameters will be saved for later. A summary of parameters with values and descriptions can be found in Table 1.

In order to choose suitable parameters for the model, the parameters p, w_{min}, w_{max}, and h were chosen to agree with observations from stadium waves. In this case, we used the width of the wave given by [FHV02], which is said to range from 15 to 20 seats. In our scaled-down representation of the stadium, this corresponds to 4 seats. The set of parameters from Table 1 is one set of parameters that produces a propagating wave 4 seats wide given a starting configuration of 4 full columns of people participating in the wave. Figure 2 illustrates the evolution of this initial wave with the chosen set of parameters. It is important to note that this set of parameters is not the only one that produces a stadium wave with these characteristics.

Parameter	Value	Description
R	50	Number of rows
N	250	Number of seats per row
p	5	Max number of seats with non-zero influence
l	2	Number of rows above/ below influencing spectator
w_{min}	0.5	Minimum weight
w_{max}	1	Maximum weight
W	19	Total number of non-zero weights per spectator
h	0.2	Wave threshold

Table 1: Model Parameters and Descriptions



(a) $t = 0$

(b) $t = 10$

(c) $t = 20$

(d) $t = 30$

Figure 2: Evolution of a full wave with the chosen set of parameters

2.2 Model

Our model is based on a directed, agent-based social network model [BP20]. The entire stadium, which is assumed to have the same number of rows and columns is represented by a matrix S defined as follows:

$$S(t) = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & x_{1,3}(t) & \cdots & x_{1,N}(t) \\ x_{2,1}(t) & x_{2,2}(t) & x_{2,3}(t) & \cdots & x_{2,N}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{R,1}(t) & x_{R,2}(t) & x_{R,3}(t) & \cdots & x_{R,N}(t) \end{bmatrix} \tag{2}$$

where spectators in the stadium are represented as variables $x_{i,j}(t)$ with $1 \leq i \leq R$ and $1 \leq j \leq N$, where r denotes the total number of rows in the stadium and N denotes the total number of people per row and t denotes time. Each variable $x_{i,j}(t)$ can take on the values 0 or 1, which encode whether the person seating at that position is raising their arms and participating in the wave ($x_{i,j}(t) = 1$) or if they are not participating ($x_{i,j}(t) = 0$).

In order to evolve the system, it is useful to convert this matrix of states into a single column vector that can be multiplied by a transition matrix between each time intervals. We let $\vec{x}(t)$ denote this vector at time t. To construct the vector, we arrange the matrix into a row vector so that column order is conserved:

$$\begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & x_{1,3}(t) & \cdots & x_{1,N}(t) \\ x_{2,1}(t) & x_{2,2}(t) & x_{2,3}(t) & \cdots & x_{2,N}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{R,1}(t) & x_{R,2}(t) & x_{R,3}(t) & \cdots & x_{R,N}(t) \end{bmatrix} \mapsto \begin{bmatrix} x_{1,1}(t) \\ \vdots \\ x_{1,N}(t) \\ x_{2,1}(t) \\ \vdots \\ x_{2,N}(t) \\ \vdots \\ x_{r,1}(t) \\ \vdots \\ x_{r,N}(t) \end{bmatrix} = \vec{x}(t) \tag{3}$$

We then construct transition matrices A, B, C to encode the weights from the interactions in the same row, one row above/below, and two rows above/below respectively. To construct these matrices, consider the vector $\vec{a} = [0\ 1\ 7/8\ 3/4\ 5/8\ 1/2\ 0\ \cdots\ 0]$ representing the weights from a particular row towards the first person in that row. Since all spectators are identical regarding the weights to other spectators around them, the matrix A can be constructed by shifting the vector $\vec{a}$ to the right and wrapping around the last element to the front.

$$A = \begin{bmatrix} 0 & 1 & 7/8 & 3/4 & 5/8 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 7/8 & 3/4 & 5/8 & 1/2 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 7/8 & 3/4 & 5/8 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 7/8 & 3/4 & 5/8 & 1/2 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 7/8 & 3/4 & 5/8 & 1/2 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{4}$$

Using the same logic, matrices B and C can be constructed in a similar manner with the vectors

$$\vec{b} = \begin{bmatrix} 0 & 1 & 5/6 & 4/6 & 3/6 & 0 & \cdots & 0 \end{bmatrix} \tag{5}$$

$$\vec{c} = \begin{bmatrix} 0 & 1 & 3/4 & 2/4 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{6}$$

The transition matrix T is then defined as a block band matrix with A in the main diagonal, B in the first lower and upper off-diagonal and C on the second lower and upper off-diagonals:

$$T = \frac{1}{W} \begin{bmatrix} A & B & C & & & & & \\ B & A & B & C & & & & \\ C & B & A & B & C & & & \\ & C & B & A & B & C & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & C & B & A & B & C \\ & & & & C & B & A & B \\ & & & & & C & B & A \end{bmatrix}. \tag{7}$$

Here W is a normalization constant that denotes the number of non-zero weights in a given row. This matrix is sparse and therefore efficient for computation. Finally, we apply a threshold function σ_h element-wise for a given threshold value h, defined as

$$\sigma_h(x_i) = \begin{cases} 0 & x_i < h \\ 1 & x_i > h \end{cases}. \tag{8}$$

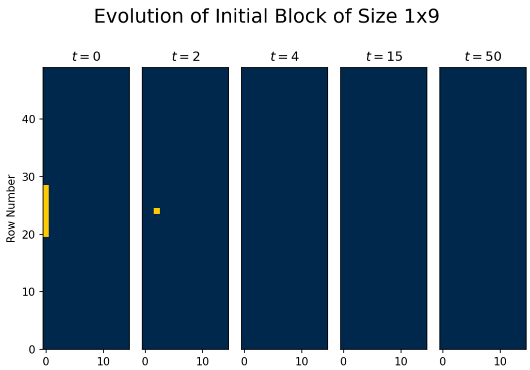
Therefore one time step evolution of the stadium is calculated as

$$\vec{x}(t+1) = \sigma_h \left(T \vec{x}(t) \right) \tag{9}$$

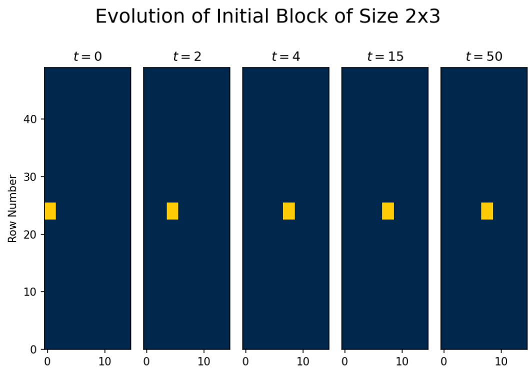
3 Presentation and Analysis of Results

3.1 Starting a Stadium Wave

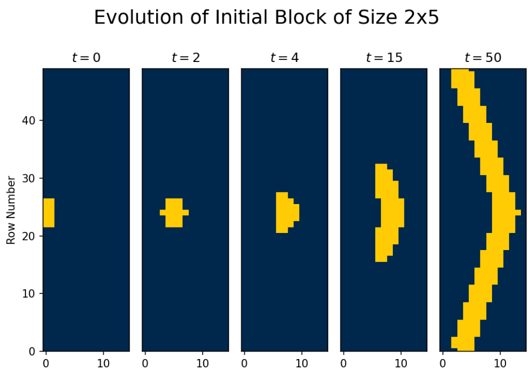
In order to investigate how the propagation and growth of the stadium wave behaves in this model with different starting arrangements, a series of different initial conditions with different sizes were tested. These initial conditions were rectangular groups of spectators centered in the middle of a column with varying width and length. Starting with a 1x1 rectangle, different width and height combinations were tested. Height varied from 1 to 9 in increments of 2, while width varied from 1 to 4 in increments of 1. Four qualitatively different initial configurations and their evolution across 50 time-steps is illustrated in Figure 3.



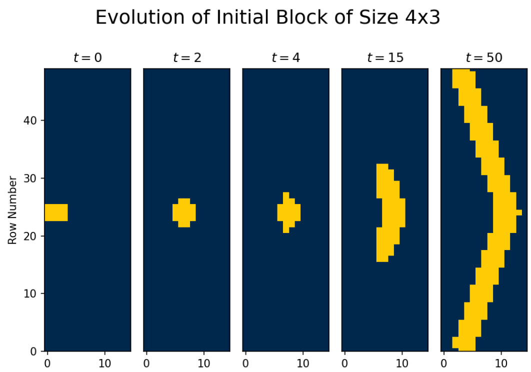
(a) Starting block of dimensions 1x9



(b) Starting block of dimensions 2x3



(c) Starting block of dimensions 2x5

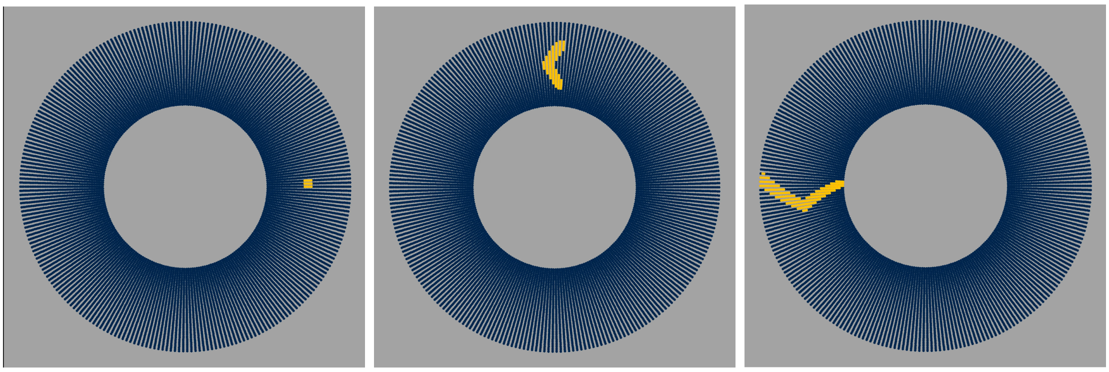


(d) Starting block of dimensions 4x3

Figure 3: Evolution of four different starting configurations

		Initial Block Width			
		1	2	3	4
Initial Block Height	1	None	None	None	None
	3	None	Propagation Only	Growth	Growth
	5	None	Growth	Growth	Growth
	7	None	Growth	Growth	Growth
	9	None	Growth	Growth	Growth

Table 2: Wave propagation as a function of initial configuration size



(a) $t = 0$ (b) $t = 21$ (c) $t = 42$

Figure 4: Evolution of a 3x5 starting block on a stadium

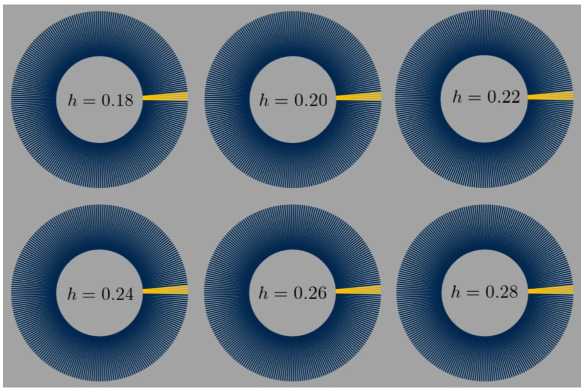
As shown, there are three distinct qualitative developments of the wave. First there is dissipation of the wave which can be observed in the 1x9 case in Figure 3a. Then there is propagation without growth, seen in the 2x3 case in Figure 3b. Finally, there is propagation with growth, illustrated in Figures 3c and 3d. Table 2 summarizes the results for all tested configurations. In addition to this, Figure 4 shows the evolution of a 3x5 block as seen on a stadium. It can be noted that in order to have a propagating wave, the initial configuration must have dimensions larger than one in both directions. That is, a single file or a single column of spectators is not enough to start a successful wave.

One interesting phenomenon that can be observed in these examples is that even though the parameters were chosen to propagate a wave that is completely straight, as seen in Figure 2, when starting with a smaller wave configuration at the center of the stadium, the wave lags towards the edges, creating a v-shaped wave. This is a result of the chosen threshold parameter h, and will be further investigated in the next section.

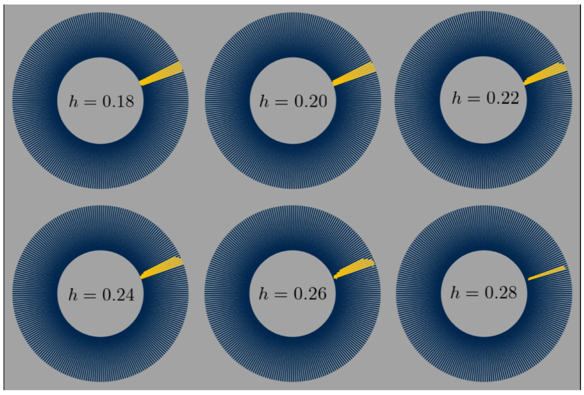
3.2 Variation of the Threshold Parameter h

We now turn to investigating the role of the threshold parameter h in the evolution of the wave, as well as the qualitative effects on the size and shape of the wave. Through experimentation, it was determined that for the chosen set of parameters and model, wave propagation occurred when the threshold h lies in the approximate range between 0.15 and 0.25. Thresholds lower than $h < 0.15$ result in propagation of the "wave" of the whole stadium, while thresholds higher than $h > 0.25$ will kill off a starting wave, causing no propagation. At the same time, as mentioned earlier, higher thresholds around $h \approx 22$ cause the propagating wave to lag towards the end, creating a v-shaped wave that propagates at approximately the same speed. Higher thresholds create a more pronounced kink in the middle of the wave. Finally, as discussed, thresholds higher than $h = 0.25$ cause the wave to decay and eventually vanish. The rate at which the wave vanishes is proportional to the threshold, i.e. higher thresholds will cause the wave to vanish at a faster rate. These results can be observed in Figure **??**.

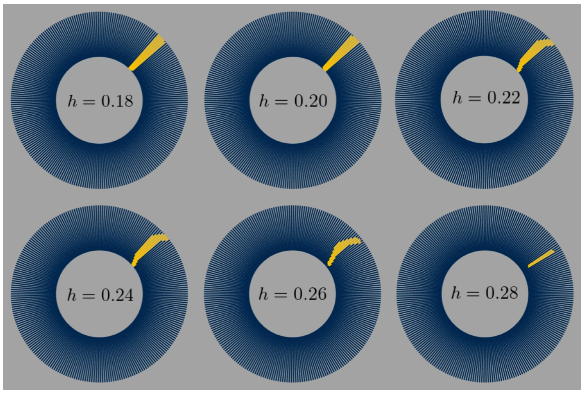
Practically, these results predict that in a stadium where everyone has on average the same tendency to stand up and participate in a wave, the shape of the wave will be influenced by this tendency to stand up. Higher tendency to participate in the wave (lower thresholds) result in a more uniform wave, while lower tendency to participate (higher thresholds) can cause the wave to deform.



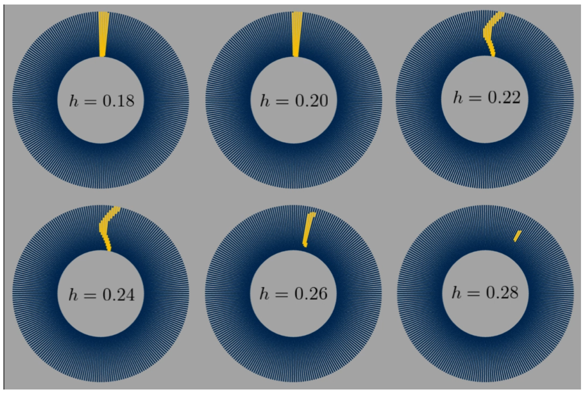
(a) $t = 0$



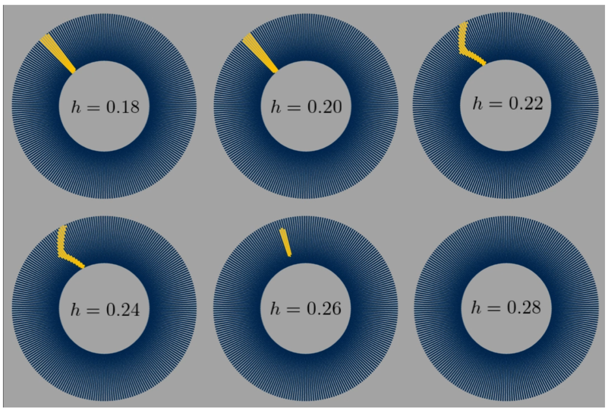
(b) $t = 5$



(c) $t = 10$



(d) $t = 20$



(e) $t = 30$

Figure 5: Evolution of stadium waves with varying thresholds from $h = 0.18$ to $h = 0.28$

4 Strengths, Limitations, and Future Work

This model accurately reproduces different kinds of stadium waves for a set of parameters. Additionally, it produces believable output regarding the propagation of this wave due to different starting configurations. Thanks to its relatively simple construction, it is very computationally friendly and easily scalable to model larger stadiums with more and more people to investigate the dynamics of those.

However, this simplicity comes as a result of some simplifying assumptions, which may not accurately represent a wave going through a sports stadium in real life. For example, it assumes that everyone is equally likely to stand up and participate. However, there are factors such as age and number of times a person has already participated in the wave that can change the likelihood of them participating. Not everyone in the stadium is as likely or eager to participate in a wave, and this model shows that waves, once started, go on. For the purposes of this model, which was to investigate the starting behavior of the wave, this is not a problem but if we were to extend this model to look at wave dynamics throughout its whole duration, this needs to be addressed. The model also assumes the same number of seats in every row and that every seat is full. Perhaps this is a good approximation for large stadiums, where the circumference is much larger than the distance from the first to the last row. However, stadiums have a decreasing number of seats per row as the stadium tapers down, and are rarely fully occupied. This alters the behaviour of the wave is ways that are not captured by this model. Finally, for simplicity we assumed a linear decrease of the weights of the people around the focused node. While this produced believable behavior, there may be other better ways of arranging the influence of spectators.

For future work, an improvement to this model would be to add these factors into the model in order to create a more accurate representation of the wave at sports stadiums. One of the things that could be done is to use a different drop in the weights of the people around the focused node such as exponentially decreasing. Another factor in real stadiums is that they have different sections. For example, there could be a student section that is more likely to stand up than the rest of the attendees. This would affect the activation threshold of that section of the stadium. Additionally, the model assumes that the influence is from people directly around the focused node, however the people across from the node and in their line of sight could also influence their decision to stand up. If someone sees people across from them participating in the wave, they will be more likely to participate in the wave when it reaches them. This idea could be incorporated into the model in the future. Finally, we can see that, contrary to what happens in stadiums, this model has a fully-fledged wave that spans every row within one quarter of a turn around the stadium. However, as any regular attendee to the Big House can attest to, it usually takes several rounds for a full wave to form. Further adjustment to the model and its parameters is required to replicate this behavior as well.

References

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