Modeling 1-D Cold Electrostatic Plasma with a Lagrangian Particle Method

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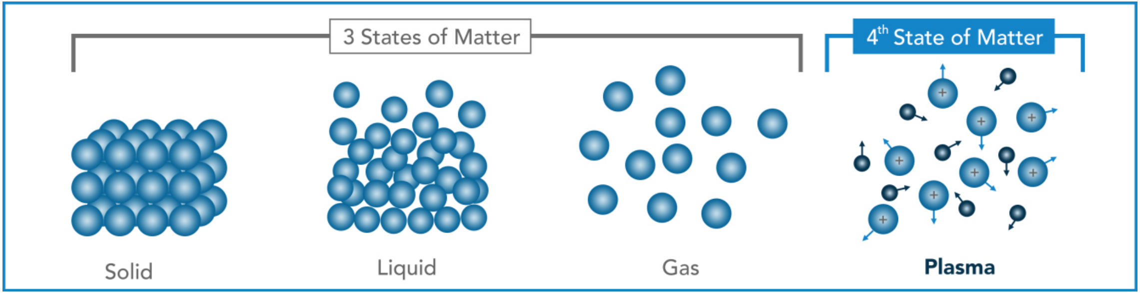
June 20, 2023

Outline

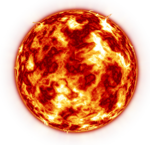
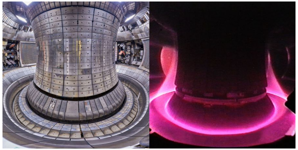
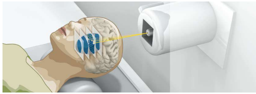
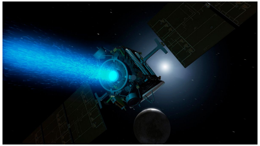
Preview

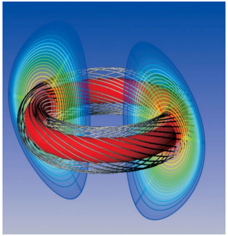
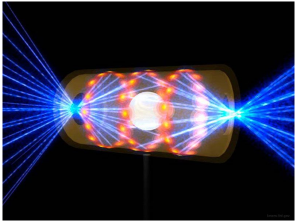
What Is Plasma?



Credit: Gaton Medical

Applications of Plasma


Plasma Distribution

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- Taken to be normalized:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, v, t)dxdv = 1$$

The Vlassov-Poisson Equations

For $(x, v) \in [0, 1] \times (-\infty, \infty)$:

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$$f(x, v, t) = f_0(x, v) \qquad \rho(x, t) = q \int_{-\infty}^{\infty} f(x, v, t) dv + 1$$

Green's Function Solution for $\phi(x,t)$

We can solve the Poisson Eq. $(-\partial_x^2\phi = \rho)$ with the periodic extension of the free-space **Green's Function** for the 1D Laplacian:

$$\phi(x,t) = \int_0^1 (g(x,y) - xy)\,\rho(y,t)dy, \quad g(x,y) = -\frac{1}{2}|x-y|$$

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From the potential ϕ we can also obtain the electric field E:

$$E(x,t) = \int_{-\infty}^{\infty} \int_0^1 \underbrace{\left(g_x(x,y) + x - y\right)}_{\text{Kernel } K(x,y)} f(y,v,t) dy dv$$

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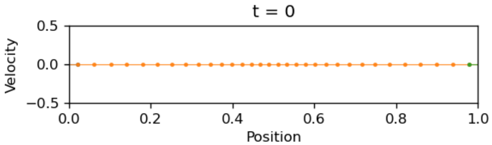
Particle Phase Space ($N = 32$, $\delta = 0.0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)



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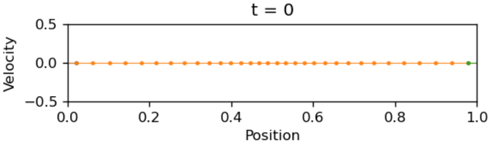

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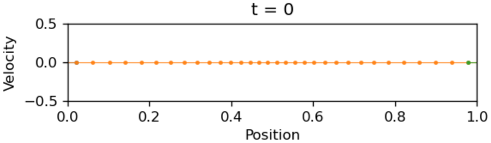

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$$E(x,t) = \int_{-\infty}^{\infty}\int_0^1 K(x,y)f(y,v,t)\,dydv$$
$$\approx \sum_{j=1}^{N} K(x,x_j)w_j$$

Evolving the Plasma

With a discretized plasma, we can calculate the acceleration felt on each of the particles:

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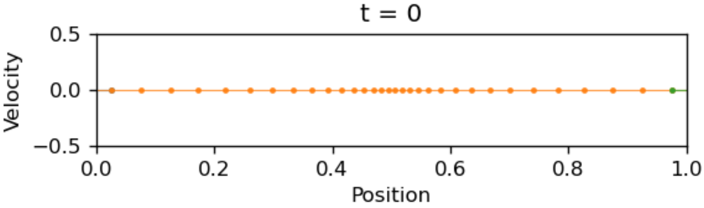
$\epsilon = 0.1$



Results

VIDEO HERE

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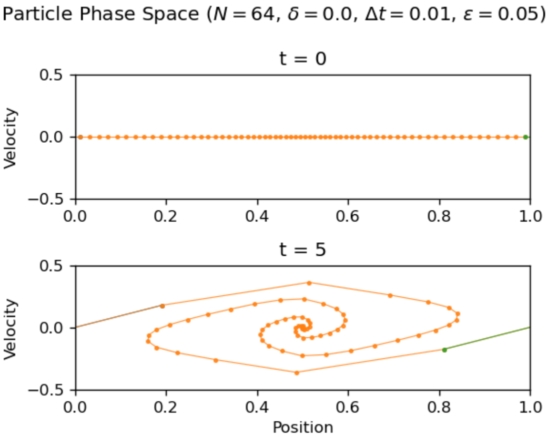
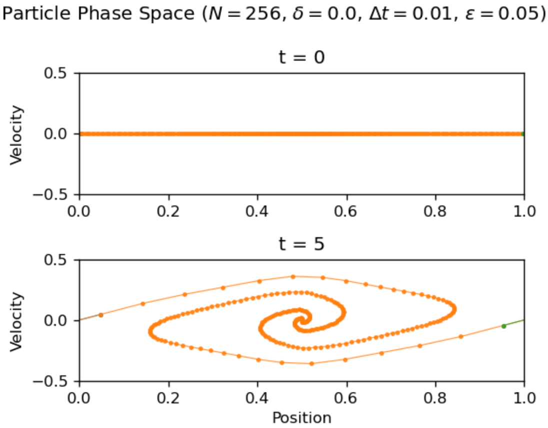
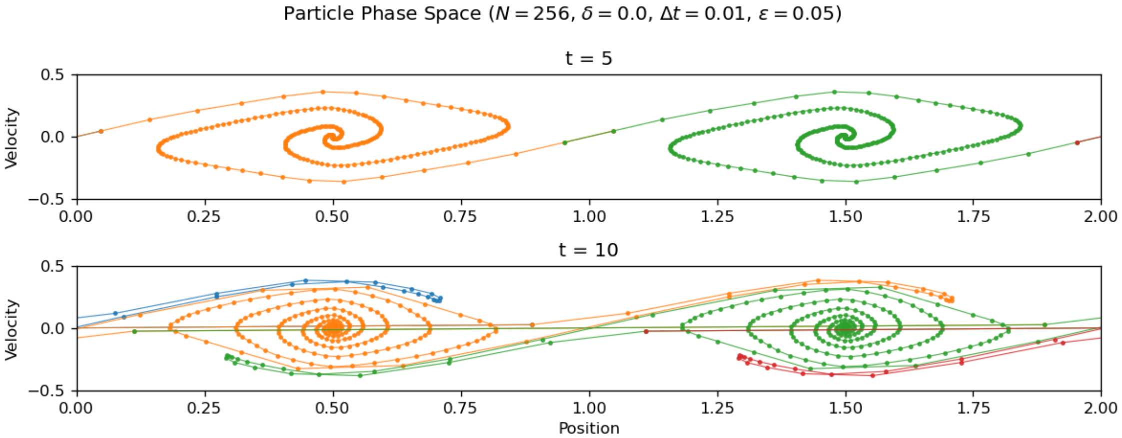

Particle Phase Space ($N = 64$, $\delta = 0.0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)
t = 0
t = 5
Velocity
Position
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0


Particle Phase Space ($N = 256$, $\delta = 0.0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)
t = 0
t = 5
Velocity
Position
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0

Results



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t = 5
Velocity
0.5
0.0
−0.5
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t = 10
Velocity
0.5
0.0
−0.5
0.00
0.25
0.50
0.75
1.00
1.25
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1.75
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Position

Some Problems

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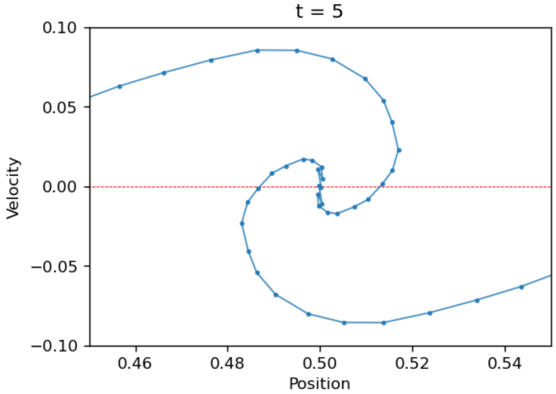
Irregularities at the center:

Particle Phase Space ($N = 128$, $\delta = 0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)



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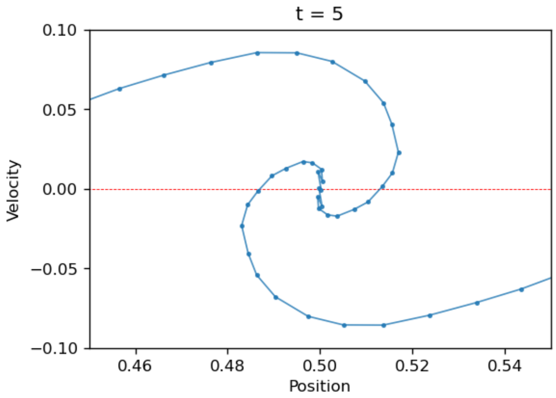
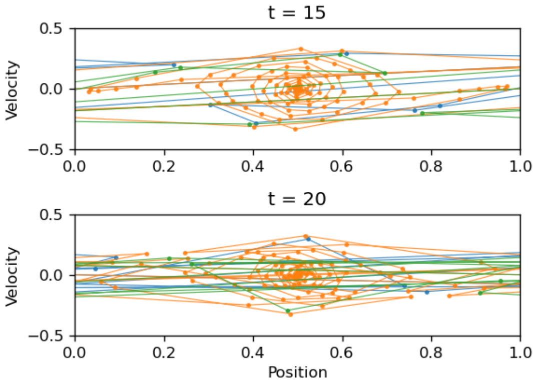


Loss of definition over time:

Particle Phase Space ($N = 128$, $\delta = 0.0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)



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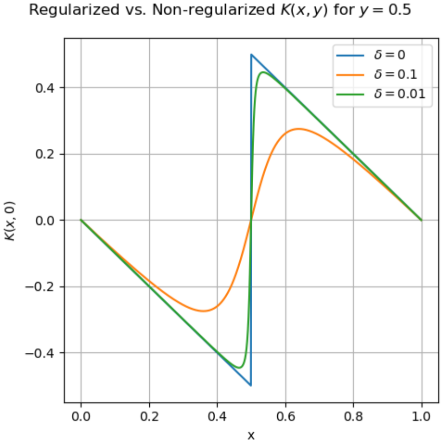
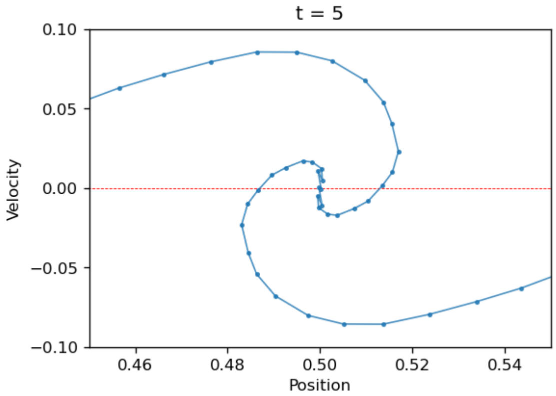


Regularized Electric Field

Particle Phase Space ($N = 128$, $\delta = 0$, $\Delta t = 0.01$, $\varepsilon = 0.05$)



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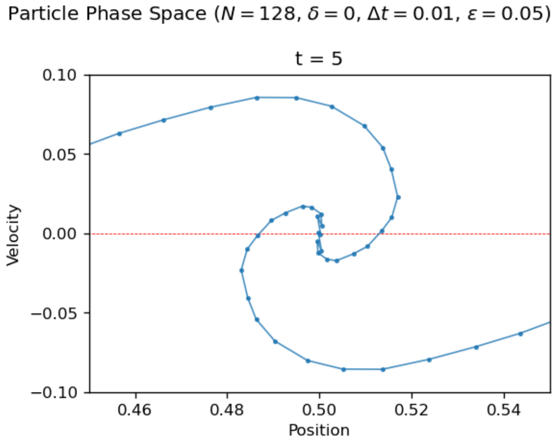

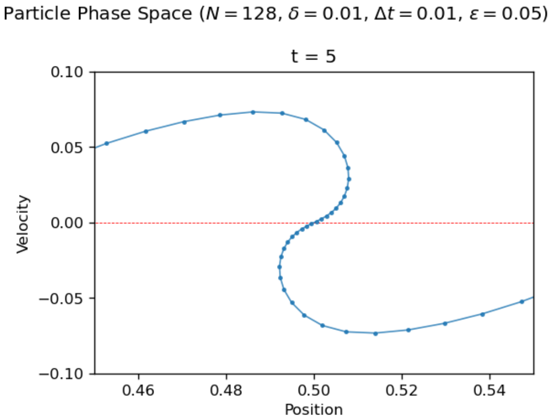

Regularized Electric Field

VIDEO OF REGULARIZED KERNEL

Further Work

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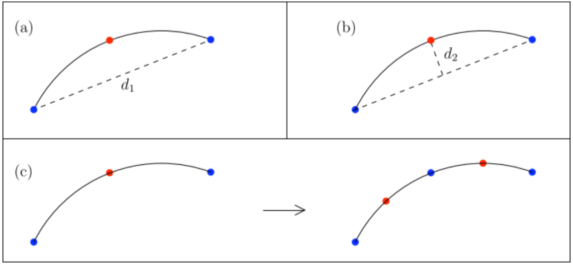
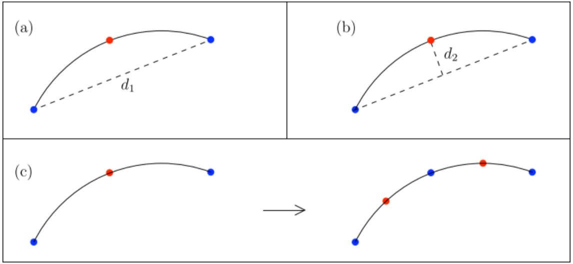
- Adaptive particle insertion



Further Work

- Adaptive particle insertion
- Evaluate 2-stream instability



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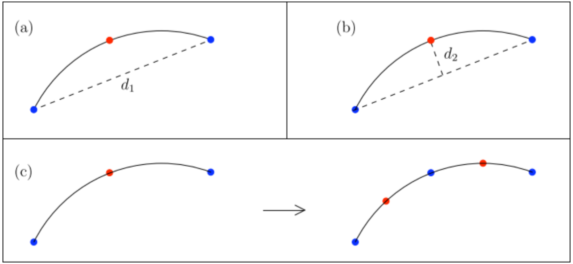
- Adaptive particle insertion
- Evaluate 2-stream instability
- Extend to warm distributions



Questions?